# ESTIMATION OF CORTICAL MULTIVARIATE AUTOREGRESSIVE MODELS FOR EEG/MEG USING AN EXPECTATION-MAXIMIZATION ALGORITHM

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## ABSTRACT

A new method for estimating multivariate autoregressive (MVAR) models of cortical connectivity from surface EEG or MEG measurements is presented. Conventional approaches to this problem first attempt to solve the inverse problem to estimate cortical signals and then fit an MVAR model to the estimated signals. Our new approach expresses the measured data in terms of a hidden state equation describing MVAR cortical signal evolution and an observation equation that relates the hidden state to the surface measurements. We develop an expectation-maximization (EM) algorithm to find maximum likelihood estimates of the MVAR model parameters. Simulations show that this one-step approach performs significantly better than the conventional two-step approach at estimating the cortical signals and detecting functional connectivity between different cortical regions.

***Index Terms***— Cortical connectivity, multivariate autoregressive model, Expectation-maximization algorithm, Granger causality, partial directed coherence

## 1. INTRODUCTION

Multivariable autoregressive (MVAR) models provide a mechanism for assessing causality in the sense of Granger [1]. They have been employed to study the functional connectivity of the brain based on invasive recordings [2] and to model interactions between signals at different scalp measurement locations [3]. Identifying cortical MVAR models from EEG/MEG data requires estimation of both the cortical signals and the corresponding model parameters. The conventional approach to this problem is to first obtain estimated cortical signals by solving the ill-conditioned EEG/MEG inverse problem and then fit a MVAR model to the estimated signals. For example, Hui and Leahy [4] use linearly constrained minimum variance beamformers to estimate the cortical signal associated with different regions of interest from the measured data. Next they solve the Yule-Walker equations formed using the estimated cortical signals to obtain the MVAR model parameters.

In this paper we formulate the cortical connectivity MVAR model estimation problem for EEG/MEG in terms of a "hidden" state equation and an observation equation that relates the hidden state to the surface measurements. The state equation describes the MVAR model relating the cortical signals and the observation equation relates the measured data to the hidden cortical signals. We obtain the maximum likelihood estimates of the MVAR model parameters using an expectation-maximization (EM) algorithm [5]. Estimates of cortical signals are obtained from the fixed interval smoother implemented in the expectation step of the EM algorithm. We use simulated data to show that this one-step approach to MVAR model identification results in improved cortical signal estimates and probability of detecting significant connectivity relative to the two-step approach of [4].

The application of the EM principle to integrated estimation of cortical signals and cortical MVAR models is new to the best of our knowledge. The improved performance obtained with the EM approach extends the range of cortical connectivity questions that can be reliably answered using EEG/MEG measurements.

## 2. STATE-SPACE MVAR MODEL FOR EEG/MEG

Let $\mathbf{y}_n^j$ and $\mathbf{x}_n^j$ be the $j^{th}$ epoch of $l \times 1$ EEG/MEG measurement data and $m \times 1$ state vector, respectively, at time $n$. The elements of $\mathbf{x}_n^j$ are the cortical signals associated with the regions in the MVAR model and may represent dipoles, multipoles, or cortical patches. We assume that $\mathbf{y}_n^j, n = 1, 2, \ldots, N, j = 1, 2, \ldots, J$ are generated by the linear time-invariant state space system

$$\mathbf{x}_n^j = \sum_{r=1}^{p} \mathbf{A}_r \mathbf{x}_{n-r}^j + \mathbf{w}_n^j \quad (1)$$

$$\mathbf{y}_n^j = \mathbf{C}\mathbf{x}_n^j + \mathbf{v}_n^j \quad (2)$$

where $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_p$ are the $m \times m$ matrices of MVAR model coefficients, $p$ is the model order, $\mathbf{C}$ is the $l \times m$

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observation matrix, and $\mathbf{w}_n^j$ and $\mathbf{v}_n^j$ are the $j^{th}$ epoch of $m \times 1$ state noise vector and $l \times 1$ observation noise vector respectively. The $m$ columns of $\mathbf{C}$ consist of the forward solutions that map the cortical signals to the $l$ EEG/MEG measurements. Define $\tilde{\mathbf{A}} = [\mathbf{A}_1\, \mathbf{A}_2\, \ldots\, \mathbf{A}_p]$ and $\tilde{\mathbf{x}}_{n-1}^j = [\mathbf{x}_{n-1}^j, \mathbf{x}_{n-2}^j, \ldots, \mathbf{x}_{n-p}^j]^T$, so Eq. (1) can be rewritten as

$$\mathbf{x}_n^j = \tilde{\mathbf{A}}\tilde{\mathbf{x}}_{n-1}^j + \mathbf{w}_n^j. \tag{3}$$

The state noise $\mathbf{w}_n^j$ and observation noise $\mathbf{v}_n^j$ are assumed to be zero-mean Gaussian vectors with covariance $E[\mathbf{w}_n^j \mathbf{w}_m^{i^T}] = \mathbf{Q}\delta_{n,m}\delta_{j,i}$ and $E[\mathbf{v}_n^j \mathbf{v}_m^{i^T}] = \mathbf{R}\delta_{n,m}\delta_{j,i}$. The $m \times 1$ initial state vector $\mathbf{x}_0^j$ is also assumed to be Gaussian distributed with unknown mean vector $\boldsymbol{\mu}_0$ and covariance matrix $\boldsymbol{\Sigma}_0$.

### 2.1. EM Algorithm for MVAR Model Estimation

The EM algorithm iteratively computes the ML estimates of $\boldsymbol{\theta} = \{\tilde{\mathbf{A}}, \mathbf{Q}, \mathbf{R}, \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0\}$ assuming $\mathbf{x}_n^j, n = 1, 2, \ldots, N, j = 1, 2, \ldots, J$ is hidden data. Define $\mathbf{X}_N^J = \{\mathbf{x}_0^1, \ldots, \mathbf{x}_N^1, \ldots, \mathbf{x}_0^J, \ldots, \mathbf{x}_N^J\}$ as the set of hidden data and $\mathbf{Y}_N^J = \{\mathbf{y}_1^1, \ldots, \mathbf{y}_N^1, \ldots, \mathbf{y}_1^J, \ldots, \mathbf{y}_N^J\}$ as the set of observed data. Together $\mathbf{X}_N^J$ and $\mathbf{Y}_N^J$ form the complete data set. The complete data likelihood function is given by

$$p(\mathbf{Y}_N^J, \mathbf{X}_N^J; \boldsymbol{\theta}) = \prod_{j=1}^{J} p(\mathbf{x}_0^j) \prod_{n=1}^{N} p(\mathbf{y}_n^j|\mathbf{x}_n^j)\; p(\mathbf{x}_n^j|\tilde{\mathbf{x}}_{n-1}^j) \tag{4}$$

where the probability densities are given by the following Gaussian distributions

$$\begin{aligned} p(\mathbf{x}_0^j) &\sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) && (5)\\ p(\mathbf{y}_n^j|\mathbf{x}_n^j) &\sim N(\mathbf{C}\mathbf{x}_n^j, \mathbf{R}) && (6)\\ p(\mathbf{x}_n^j|\tilde{\mathbf{x}}_{n-1}^j) &\sim N(\tilde{\mathbf{A}}\tilde{\mathbf{x}}_{n-1}^j, \mathbf{Q}) && (7) \end{aligned}$$

Thus the log-likelihood function for the complete data is

$$\begin{aligned} &\log p(\mathbf{Y}_N^J, \mathbf{X}_N^J; \boldsymbol{\theta}) = \\ &\quad -\frac{J}{2}\log|\boldsymbol{\Sigma}_0| - \frac{1}{2}\sum_{j=1}^{J}(\mathbf{x}_0^j - \boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}_0^{-1}(\mathbf{x}_0^j - \boldsymbol{\mu}_0) \\ &\quad -\frac{JN}{2}\log|\boldsymbol{Q}| \\ &\quad -\frac{1}{2}\sum_{j=1}^{J}\sum_{n=1}^{N}(\mathbf{x}_n^j - \tilde{\mathbf{A}}\tilde{\mathbf{x}}_{n-1}^j)^T\mathbf{Q}^{-1}(\mathbf{x}_n^j - \tilde{\mathbf{A}}\tilde{\mathbf{x}}_{n-1}^j) \\ &\quad -\frac{JN}{2}\log|\boldsymbol{R}| \\ &\quad -\frac{1}{2}\sum_{j=1}^{J}\sum_{n=1}^{N}(\mathbf{y}_n^j - \mathbf{C}\mathbf{x}_n^j)^T\mathbf{R}^{-1}(\mathbf{y}_n^j - \mathbf{C}\mathbf{x}_n^j) \\ &\quad -\frac{J(mp + N(mp+l))}{2}\log 2\pi \end{aligned} \tag{8}$$

The EM algorithm iteratively maximizes the conditional expectation of the log-likelihood of the complete data with respect to the unknown parameters in $\boldsymbol{\theta}$ using two steps at each iteration. Let $\boldsymbol{\theta}^{[i]}$ represent the estimates from the $i^{th}$ iteration. In the expectation or E-step of the $(i+1)^{th}$ iteration $q(\boldsymbol{\theta}|\boldsymbol{\theta}^{[i]})$ is calculated as

$$\begin{aligned} q(\boldsymbol{\theta}|\boldsymbol{\theta}^{[i]}) &= E[\log p(\mathbf{Y}_N^J, \mathbf{X}_N^J; \boldsymbol{\theta})|\mathbf{Y}_N^J, \boldsymbol{\theta}^{[i]}] \\ &= -\frac{J}{2}\log|\boldsymbol{\Sigma}_0| \\ &\quad -\frac{1}{2}\mathrm{tr}\left\{\boldsymbol{\Sigma}_0^{-1}\sum_{j=1}^{J}\Big(\mathbf{P}_{0|N}^j \right. \\ &\quad \left. +(\mathbf{x}_{0|N}^j - \boldsymbol{\mu}_0)(\mathbf{x}_{0|N}^j - \boldsymbol{\mu}_0)^T\Big)\right\} \\ &\quad -\frac{JN}{2}\log|\mathbf{Q}| \\ &\quad -\frac{1}{2}\mathrm{tr}\left\{\mathbf{Q}^{-1}(\mathbf{D} - \mathbf{E}\tilde{\mathbf{A}}^T - \tilde{\mathbf{A}}\mathbf{E}^T + \tilde{\mathbf{A}}\mathbf{F}\tilde{\mathbf{A}}^T)\right\} \\ &\quad -\frac{JN}{2}\log|\mathbf{R}| \\ &\quad -\frac{1}{2}\mathrm{tr}\left\{\mathbf{R}^{-1}\sum_{j=1}^{J}\sum_{n=1}^{N}\Big[(\mathbf{y}_n^j - \mathbf{C}\mathbf{x}_{n|N}^j)\times \right. \\ &\quad \left. (\mathbf{y}_n^j - \mathbf{C}\mathbf{x}_{n|N}^j)^T + \mathbf{C}\mathbf{P}_{n|N}^j\mathbf{C}^T\Big]\right\} \\ &\quad -\frac{J(mp + N(mp+l))}{2}\log 2\pi \end{aligned} \tag{9}$$

where $\mathbf{x}_{n|N}^j$ is the fixed interval smoother estimate of $\mathbf{x}_n^j$, $\mathbf{P}_{n|N}^j$ is the estimation error covariance matrix, and

$$\begin{aligned} \mathbf{D} &= \sum_{j=1}^{J}\sum_{n=1}^{N}\mathbf{P}_{n|N}^j + \mathbf{x}_{n|N}^j\mathbf{x}_{n|N}^{j^T} && (10)\\ \mathbf{E} &= \sum_{j=1}^{J}\sum_{n=1}^{N}\mathbf{P}_{n,n-1|N}^j + \mathbf{x}_{n|N}^j\tilde{\mathbf{x}}_{n-1|N}^{j^T} && (11)\\ \mathbf{F} &= \sum_{j=1}^{J}\sum_{n=1}^{N}\mathbf{P}_{n-1|N}^j + \tilde{\mathbf{x}}_{n-1|N}^j\tilde{\mathbf{x}}_{n-1|N}^{j^T}. && (12) \end{aligned}$$

Here $\mathbf{P}_{n,n-1|N}^j$ is the one-lag cross covariance and is also obtained from the fixed interval smoother [6].

The second step of the EM iteration is the maximization or M-step. The M-step computes the new estimate $\boldsymbol{\theta}^{[i+1]}$ by maximizing $q(\boldsymbol{\theta}|\boldsymbol{\theta}^{[i]})$ with respect to $\boldsymbol{\theta}$ to obtain

$$\tilde{\mathbf{A}}^{[i+1]} = \mathbf{E}\mathbf{F}^{-1} \quad (13)$$

$$\mathbf{Q}^{[i+1]} = \frac{1}{JN}(\mathbf{D} - \mathbf{E}\mathbf{F}^{-1}\mathbf{E}^T) \quad (14)$$

$$\mathbf{R}^{[i+1]} = \frac{1}{JN}\sum_{j=1}^{J}\sum_{n=1}^{N}[(\mathbf{y}_n^j - \mathbf{C}\mathbf{x}_{n|N}^j) \times (\mathbf{y}_n^j - \mathbf{C}\mathbf{x}_{n|N}^j)^T + \mathbf{C}\mathbf{P}_{n|N}^j\mathbf{C}^T] \quad (15)$$

$$\boldsymbol{\mu}_0^{[i+1]} = \frac{1}{J}\sum_{j=1}^{J}\mathbf{x}_{0|N}^j \quad (16)$$

$$\boldsymbol{\Sigma}_0^{[i+1]} = \frac{1}{J}\sum_{j=1}^{J}\mathbf{P}_{0|N}^j. \quad (17)$$

The E- and M-steps are repeated until the data likelihood converges. This procedure increases likelihood at each iteration and thus is guaranteed to converge to a local maximum.

### 2.2. Estimation of Cortical Connectivity

Several connectivity metrics may be extracted from the MVAR model parameters, including Granger causality [1] and the directed transfer function (DTF) [3]. In the simulations we use the partial directed coherence (PDC) [2]

$$PDC_{ij}(f) = \frac{|\bar{A}_{ij}(f)|}{\sqrt{\sum_k \bar{A}_{kj}^*(f)\bar{A}_{kj}(f)}} \quad (18)$$

where $\bar{A}_{ij}$ is the $ij^{th}$ element of $\bar{\mathbf{A}}(f) = \mathbf{I} - \mathbf{A}(f)$, $*$ denotes the complex conjugate operator and

$$\mathbf{A}(f) = \sum_{r=1}^{p}\mathbf{A}_r e^{-j2\pi fr}. \quad (19)$$

$PDC_{ij}(f)$ represents the influence from the $j^{th}$ cortical signal to the $i^{th}$ one at frequency $f$.

### 2.3. Test for Significance

The method of surrogate data proposed by Theiler *et al.*, [7] is used to identify thresholds for testing the significance of the estimated PDC or other connectivity metric. The goal is to approximate the distribution of the estimated metric when the true connectivity is zero in order to identify a threshold resulting in a desired probability of false positive decision. This is accomplished by randomly perturbing the data to intentionally destroy any connectivity that may be present. The estimated cortical signals are transformed to the frequency domain using the FFT, the phase is randomized using independent random variables uniformly distributed on $[0, 2\pi]$, and an inverse FFT converts the phase randomized cortical signals back to the time domain. This procedure destroys any cross-correlation between cortical signals while preserving the power spectrum of each signal. The resulting MVAR and PDC estimates correspond to a network of independent or unconnected cortical regions. The randomization procedure is repeated multiple times to estimate the distribution of the metric(s) of interest under the assumption of independent cortical signals.

## 3. SIMULATION RESULTS

A brain network is simulated based on the Wernicke - Geschwind model [8] of language processing and speech production. Dipolar sources are placed in the primary auditory cortex (region 1), Wernicke's area (region 2), Broca's area (region 3), and the motor cortex (region 4). Time series are generated for each region according to an MVAR model of order $p = 2$. The MVAR coefficients are chosen to place a spectral peak near 15 Hz in region 1 and the following causal influences between regions: $1 \to 2$, $2 \to 3$, and $3 \to 4$. All other causal influences in the model are set to zero. The state noise covariance is set to $\mathbf{Q} = \frac{1}{2}\mathbf{I}$. MEG forward solutions for the dipolar sources are evaluated at 54 sensor locations uniformly distributed around the head. Observation noise is added to the simulated measured signals to obtain a desired SNR, defined as $\mathrm{tr}\{\sum_{j=1}^{J}\sum_{n=1}^{N}(\mathbf{C}\mathbf{x}_n^j)^T(\mathbf{C}\mathbf{x}_n^j)\}/\mathrm{tr}\{\sum_{j=1}^{J}\sum_{n=1}^{N}(\mathbf{v}_n^j)^T(\mathbf{v}_n^j)\}$. The spatial covariance of the noise is chosen to match that measured from a human subject in the absence of stimulus. A sampling frequency of 100 Hz is assumed. We set $N = 256$ and $J = 1$.

The linearly constrained beamforming approach (LCB) of [4] is compared to the maximum likelihood based EM approach (MLEM) described in Section 2 with respect to estimation of cortical signals and probability of detecting causal influence between regions. MLEM uses the posterior mean $\mathbf{x}_{n|N}^j$ for the estimated cortical signals. Both LCB and MLEM assume the correct MVAR model order of $p = 2$. Figure 1 depicts the normalized mean square error (NMSE) associated with LCB and MLEM estimates of the four cortical signals as a function of SNR. The NMSE is calculated by averaging over 100 independent trials. The MLEM approach results in consistently lower NMSE than does LCB over a wide range of SNRs. The mean LCB and MLEM PDC estimates for SNR = 0 dB are compared to the true PDC in Figure 2. The mean PDC from MLEM better approximates the true PDC for the non-zero causal influence ($1 \to 2$, $2 \to 3$ and $3 \to 4$).

We evaluate the probability of detecting causal influence by integrating the PDC from 12 Hz to 30 Hz. The value of the integrated PDC associated with $1 \to 2$ and $3 \to 4$ is tested for significance by comparing it to thresholds chosen to set the probability of false detection at 0.01, 0.05, 0.1, 0.25, and 0.5. The thresholds are chosen using 250 surrogate data sets and the probability of detection is evaluated using 100 indepen-

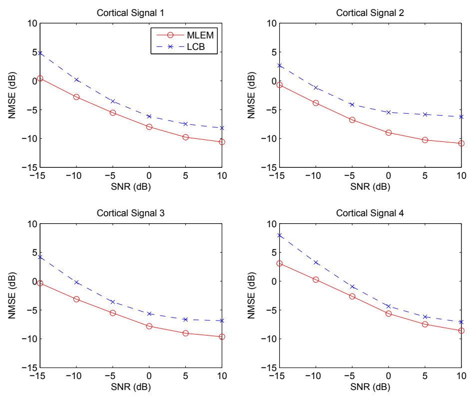


**Fig. 1**. NMSE for the four estimated cortical signals as a function of SNR. LCB: dashed blue line, MLEM: solid red line.

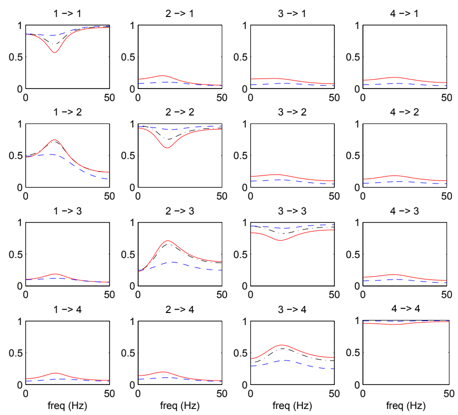


**Fig. 2**. True (dash-dotted black line) and mean estimated PDCs at SNR = 0 dB. LCB: dashed blue line, MLEM: solid red line.

dent trials. Figure 3 depicts the resulting receiver operating characteristics (ROCs) for LCB and MLEM at SNR = -15 dB. Note that the MLEM approach offers improved detection performance at all probabilities of false detection. The difference is greatest when the true PDC is weakest ($3 \rightarrow 4$).

## 4. SUMMARY

The state/observation equation formulation of MVAR cortical networks with surface EEG/MEG data leads to an EM algorithm for finding maximum likelihood estimates of MVAR model parameters and the corresponding cortical signal estimates. Simulations indicate this integrated approach provides potential performance improvements compared to a two-step method that first estimates cortical signals and then fits an MVAR model to the estimated signals.

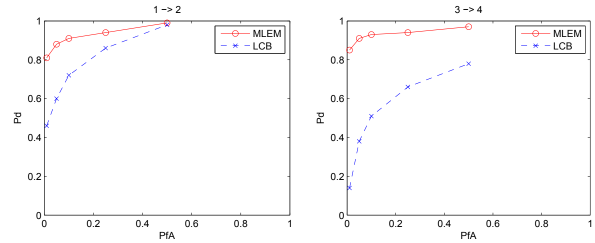


**Fig. 3**. Probability of correct detection vs probability of false detection for connections $1 \rightarrow 2$ and $3 \rightarrow 4$ at SNR = -15 dB. LCB: dashed blue line, MLEM: solid red line.

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